Technical Notes

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Thermal Postbuckling Behavior of Tapered Columns

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Introduction

MODERN aerospace structural components are generally subjected to severe thermal loading that must be carefully considered in static, vibration, and stability behavioral studies for an efficient design. In order to obtain an effective use of the material, consideration of the nonlinear behavior of the structural components is important. Further, tapered configurations contribute to the optimum utilization of the material. In this Note, these three aspects are considered with respect to columns.

This study is useful in determining the inter-rivet spacing of stiffeners in heat shields and the interstages of launch vehicles and the design adequacy of the supporting members of structural components in satellites and large space structures exposed to thermal loads.

Postbuckling solutions for uniform columns subjected to mechanical loading, wherein the nonlinearity is present through the curvature terms, is studied in Ref. 1. However, postbuckling solutions for a tapered column under thermal loading has not received much attention. In this case, since the ends of the column are restrained to move along the axis, the nonlinearity effects are introduced through axial strain.

Recently, the authors have presented a finite element formulation[2,3] for the study of the postbuckling behavior of circular plates and have also investigated the thermal postbuckling behavior of uniform columns.[4] However, whenever possible, the closed-form solutions are always attractive and time saving, particularly when parametric studies (such as the taper parameter) are involved.

This Note considers the postbuckling behavior of simply supported tapered columns of rectangular cross sections under thermal loading through a Rayleigh-Ritz analysis. Trigonometric and polynomial expressions are used for the displacements and both breadth and depth tapers are studied. The linear thermal buckling loads and the ratios of the nonlinear thermal load to the linear thermal buckling load are obtained in closed form. A few numerical results are presented.

Formulation

Figure 1 shows the configurations of the two types of tapered columns considered here. Consider a column of length $2l$ to be heated to a temperature T, which causes a stretching of the column. The stretching u is given by

$$u = 2l\alpha T \tag{1}$$

where α is the coefficient of thermal expansion. Also the axial stretching in terms of the axial force P developed due to thermal loading is obtained from

$$u = \int_{-l}^{l} \frac{P}{AE}\,\mathrm{d}x \tag{2}$$

where x is the axial coordinate, A the area of cross section, and E Young's modulus. Considering the axial force P to be constant throughout the length of the beam, we obtain from Eqs. (1) and (2)

$$P = 2l\alpha T \Big/ \int_{-l}^{l} \frac{\mathrm{d}x}{AE} \tag{3}$$

For both breadth- and depth-tapered columns, the area of cross section A varies as

$$A = A_c[1-\beta(x/l)] \quad \text{for} \quad 0 \le x \le l$$

and

$$A = A_c[1+\beta(x/l)] \quad \text{for} \quad -l \le x \le 0 \tag{4}$$

where A_c is the area of cross section at the center of the beam ($x=0$) and β the taper parameter defined as

$$\text{Breadth taper:} \quad \beta = \frac{b_c - b_e}{b_e}$$

$$\text{Depth taper:} \quad \beta = \frac{d_c - d_e}{d_e} \tag{5}$$

where b and d are the breadth and depth of the beam and subscripts c and e indicate the center and the end of the beam, respectively.

From Eqs. (3) and (4) after integration, we obtain

$$P = \frac{A_c E \alpha T \beta}{\ell n(1-\beta)} \tag{6}$$

for both cases of taper [of course, with definitions of β as in Eq. (5)]. When uniform columns ($\beta=0$, $A=A_c=A_e$) are considered, this expression is of the form

$$P = AE\alpha T \tag{6a}$$

The strain energy U of the column is

$$U = \frac{1}{2}\int_{-l}^{l} (EA\epsilon_x^2 + EI\chi_x^2)\,\mathrm{d}x \tag{7}$$

where ϵ_x and χ_x are the axial strain and curvature, respectively, given by

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \chi_x = -\frac{\partial^2 w}{\partial x^2} \tag{8}$$

where w is the transverse displacement and the moment of inertia I varies as

$$\text{Breadth taper:} \quad I = I_c\left(1 - \frac{\beta_x}{l}\right), \quad 0 \le x \le l$$

$$I = I_c\left(1 + \frac{\beta_x}{l}\right), \quad -l \le x \le 0 \tag{9}$$

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Depth taper: $I = I_c\left(1 - \frac{\beta_x}{l}\right)^3, \quad 0 \le x \le l$

$$I = I_c\left(1 + \frac{\beta_x}{l}\right)^3, \quad -l \le x \le 0 \tag{10}$$

where I_c is the moment of inertia at the center of the beam.

The work W done by the thermal load is given by

$$W = \frac{P}{2}\int_{-l}^{l} w'^2 \mathrm{d}x \tag{11}$$

where the prime denotes differentiation with respect to x.

Solutions

The buckling and postbuckling characteristics of the columns can be obtained from U and W by employing a Rayleigh-Ritz scheme.

For simply supported columns, the displacement distributions of the type,

Trigonometric: $u = a\sin(\pi x/l)$

$$w = b\cos(\pi x/2l) \tag{12}$$

Polynomial: $u = a(x/l)\,[1 - (x/l)^2]$

$$w = b[1 - (6/5)(x/l)^2 + (1/5)(x/l)^4] \tag{13}$$

satisfy all of the geometric boundary conditions and are compatible, with a and b being the constant amplitudes. The same sets of distributions are used for both the depth- and breadth-tapered cases. After substitution of Eq. (12) or (13) into Eqs. (7) and (11) and performing the integrations, a system of simultaneous algebraic equations is obtained by the Rayleigh-Ritz method as

$$\frac{\partial}{\partial a}(U - W) = 0 \tag{14}$$

and

$$\frac{\partial}{\partial b}(U - W) = 0 \tag{15}$$

The solutions of Eqs. (14) and (15) yield the linear thermal buckling load $\lambda_L\,(=\alpha T\ell^2/r_c^2)$ when the nonlinear terms are neglected and the thermal load ratio $\gamma(\lambda_{NL}/\lambda_L)$ when nonlinear terms are considered. (r_c is the radius of gyration at the center of the beam.)

The various expressions of λ_L and γ for breadth- and depth-tapered columns for both trigonometric and polynomial displacement distributions are obtained by the above procedure and are given in Table 1.

For uniform beams ($\beta = 0$), the λ_L and γ expressions given in Table 1 reduce (from both the breadth- and depth-tapered solutions) to

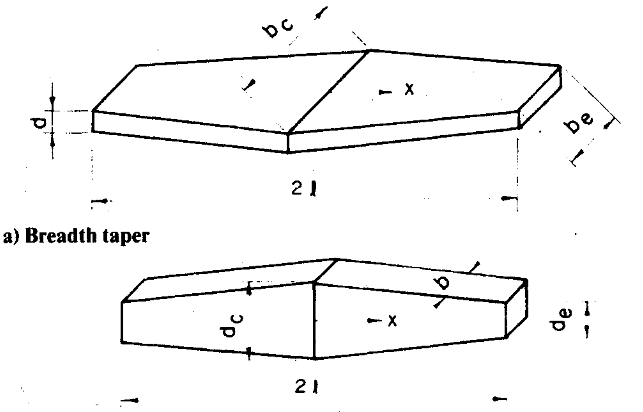
Table 1 Simply supported tapered columns: thermal radial load-amplitude relationship

	λ_{NL}/λ_L	λ	μ
Breadth-tapered columns[a]			
$\lambda_L = -\frac{\pi^2}{4}\left[1 - \frac{\beta}{2} + \frac{2\beta}{\pi^2}\right]\frac{\ln(1-\beta)}{\beta}$	$1 + \frac{1}{4}\left(\frac{b}{r_c}\right)^2\frac{1+\lambda}{1+\mu}$	$-\beta + \beta^2\left(\frac{1}{4} - \frac{8}{\pi^4}\right)$	$-\beta\left(1 - \frac{2}{\pi^2}\right) + \beta^2\left(\frac{1}{4} - \frac{1}{\pi^2}\right)$
Breadth-tapered columns[b]			
$\lambda_L = -\frac{315}{68}\left(\frac{8}{15} - \frac{\beta}{6}\right)\frac{\ln(1-\beta)}{\beta}$	$1 + 0.2600\left(\frac{b}{r_c}\right)^2\frac{1+\lambda}{1+\mu}$	$-1.1511\beta + 0.2741\beta^2$	$-0.9375\beta + 0.1953\beta^2$
Depth-tapered columns[a]			
$\lambda_L = -\frac{\pi^2}{4}\left[1 + 3\beta\left(-\frac{1}{2} + \frac{2}{\pi^2}\right) + 3\beta^2\left(\frac{1}{3} - \frac{2}{\pi^2}\right) + \beta^3\left(-\frac{1}{4} + \frac{3}{\pi^2} - \frac{12}{\pi^4}\right)\right]\frac{\ln(1-\beta)}{\beta}$	$1 + \frac{1}{4}\left(\frac{b}{r_c}\right)^2\frac{1+\lambda}{1+\mu}$	$-\beta + \beta^2\left(\frac{1}{4} - \frac{8}{\pi^4}\right)$	$-\beta\left(2 - \frac{6}{\pi^2}\right) + \beta^2\left(\frac{7}{4} - \frac{9}{\pi^2}\right) + \beta^3\left(-\frac{3}{4} + \frac{6}{\pi^2} - \frac{12}{\pi^4}\right) + \beta^4\left(\frac{1}{8} - \frac{3}{2\pi^2} + \frac{6}{\pi^4}\right)$
Depth-tapered columns[b]			
$\lambda_L = -\frac{315}{68}\left[\frac{8}{15} - \frac{\beta}{2} + \frac{8\beta^2}{35} - \frac{\beta^3}{24}\right]\frac{\ln(1-\beta)}{\beta}$	$1 + 0.2600\left(\frac{b}{r_c}\right)^2\frac{1+\lambda}{1+\mu}$	$-1.1511\beta + 0.2741\beta^2$	$-1.5624\beta + 1.0145\beta^2 - 0.3460\beta^3 + 0.0488\beta^4$

[a] With $u = a\sin\pi x/l$ and $w = b\cos\pi x/2l$. [b] With $u = a(x/l)\,[1 - (x/l)^2]$ and $w = b[1 - (6/5)(x/l)^2 + 1/5(x/l)^4]$.

Table 2 λ_L and λ_{NL}/λ_L values of tapered columns

	λ_{NL}/λ_L											
	Breadth taper						Depth taper					
Amplitude	0.0		0.2		0.4		0.0		0.2		0.4	
ratio, b/r_c	Trigo.	Polyno.	Trigo.	Polyno.	Trigo.	Polyno.	Trigo.	Polyno.	Trigo.	Polyno.	Trigo.	Polyno.
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.0100	1.0104	1.0476	1.0495	1.0445	1.0462	1.0100	1.0104	1.0536	1.0560	1.0558	1.0587
0.4	1.0400	1.0416	1.0953	1.0990	1.0889	1.0924	1.0400	1.0416	1.1071	1.1119	1.1117	1.1175
0.6	1.0900	1.0996	1.1430	1.1484	1.1334	1.1387	1.0900	1.0996	1.1607	1.1679	1.1676	1.1762
0.8	1.1600	1.1664	1.1906	1.1979	1.1779	1.1849	1.1600	1.1664	1.2142	1.2338	1.2234	1.2349
1.0	1.2500	1.2600	1.2382	1.2474	1.2223	1.2311	1.2500	1.2600	1.2678	1.2798	1.2793	1.2936
λ_L	2.4674	2.4706	2.5892	2.5842	2.7762	2.7607	2.4674	2.4706	2.3034	2.2852	2.2104	2.1725



a) Breadth taper

b) Depth taper

Fig. 1 Geometry of tapered columns.

Trigonometric: $\lambda_L = \pi^2/4 = 2.4674$

$$\gamma = 1 + 1/4\,(b/r_c)^2 \tag{16}$$

Algebraic: $\lambda_L = 42/17 = 2.4706$

$$\gamma = 1 + 2.2600(b/r_c)^2 \tag{17}$$

where r_c is the radius of gyration at the center of the beam. Note that λ_L is defined with respect to ℓ, the half length of the beam.

Numerical Results

The results for the uniform beam given in Eqs. (16) and (17) can be seen to be in good agreement when trigonometric and algebraic functions are used for the displacements. These results are also in good agreement with the finite element results presented earlier.[4]

Table 2 presents λ_L and γ values for both breadth- and depth-tapered columns for three values of β (0.0, 0.2, and 0.4). Again, the results from trigonometric and polynomial distributions are in close agreement. Further, it can be seen that for breadth-tapered columns, the linear buckling load decreases with increasing taper. The effect of nonlinearity (from γ values) can be seen to be relatively greater in the case of depth-tapered columns and it becomes pronounced when large depth tapers are considered. Also, the effect of nonlinearity decreases with increasing breadth taper, whereas the effect increases with increasing depth taper.

Conclusions

Rayleigh-Ritz solutions are presented for the linear thermal buckling load and thermal load ratios in the postbuckling range for simply supported tapered columns under thermal loading. The effect of nonlinearity on the thermal load ratios is found to be of the opposite tendency with increasing taper values in the breadth- and depth-tapered cases.

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Thermal Constriction Resistance with Arbitrary Heating in a Convectively Cooled Plate

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Nomenclature

a	= contact length
a_q	= thermal diffusivity
b	= length of the plate
Bi	= Biot number (hb/k)
c	= thickness of the plate
e	= eccentricity
Fo	= Fourier number $(a_q t/b^2)$
h	= heat-transfer coefficient
k	= coefficient of thermal conductivity
L	= characteristic length
P	= parameter defined by Eq. (18)
q	= heat flux
q^*	= dimensionless heat flux
q_0	= constant value of dimensionless heat flux
R	= thermal constriction resistance defined by Eq. (6)
R_c^*	= dimensionless thermal constriction resistance
T	= temperature distribution
T_f	= ambient temperature
x,y	= Cartesian coordinate system
α	= aspect ratio (c/b)
ϵ	= parameter defined by Eq. (21)
θ	= dimensionless temperature $(T-T_b)/T_b$
θ_0	= initial value of dimensionless temperature
μ	= characteristic root defined by Eq. (12)
ω	= parameter defined by Eq. (21)

I. Introduction

TEMPERATURE control is one of the most important design problems in achieving high reliability in a launch vehicle or spacecraft. It requires a precise heat-transfer calculation and the designer is often called upon to predict the total thermal resistance present in the available heat path from the source to the sink. The thermal resistance developed through a constriction in the heat flow is usually high when compared to the resistance away from the constriction. The accurate prediction of thermal resistance requires a detailed transient analysis of a two-dimensional constriction. However, no such analysis has been reported in the literature. In contrast, steady-state analysis has been attempted by several authors. Veziroglu and Chandra[1] have provided a theoretical and experimental study of the subject for both symmetrical and eccentric constrictions. Schneider, Yovanovich, and Cane[2] have recently provided a steady-state analysis for a planar constriction with an arbitrary heat flux boundary condition. Mehta and Bose[3] have obtained the expression for the thermal constriction resistance of a large circular plate with the heat flux provided on a disk area.

The present work develops a theory for predicting the thermal constriction resistance of an eccentric contact area subjected to an arbitrary heat flux. A detailed transient

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